**On Sufficient Conditions for Trapped Surfaces in Spherically Symmetric Spacetimes**

**Abstract.** Black hole formation in a spacetime, while not fully understood, can be indicated by the local existence of a trapped surface. The Trapped Surface Conjecture (TSC) states that trapped surfaces will form from sufficiently high concentrations of matter. Progress on the TSC has only been made using several spacetime constraints, including spatial symmetries, time symmetry, and the maximal hypersurface. We investigated the TSC in spherical symmetry by utilizing non-positivity of null expansion of a trapped surface within the context of expressions for various geometrical quantities. We found that if certain inequalities involving energy density, proper radius, and proper area of a two-surface on a time-symmetric time slice are satisfied, then a trapped surface exists inside the two-surface. These criteria predict trapped surfaces in some spacetimes, including the constant-density star, where previous analyses fail to. In the non-time-symmetric case, we found that if a two-surface on a time slice satisfies constraints involving mean curvature, mass content, radial flow, and proper radius, then a trapped surface exists inside the two-surface. This generalizes the main non-time-symmetric result obtained by Bizon et al. 1988, removing the requirement for a maximal hypersurface. Our results constitute progress toward proving the TSC and understanding black hole formation.

# 1 Introduction

In general relativity, a feature of interest in any spacetime is the possibility of gravitational collapse and black hole formation. The presence of a black hole can be indicated by the detection of its event horizon. However, predicting the existence of an event horizon is only possible by considering the entire causal structure of the spacetime, and is not possible through the consideration of local geometries alone (Faraoni, 2013). Penrose formulated the concept of a trapped surface on a local geometry of a spacetime to address this problem (Penrose, 1965). On a time slice of the spacetime, a trapped surface is a closed two-surface satisfying the condition that both sets of future-directed null normal vectors on the surface are convergent (Senovilla, 2011). The matter inside a trapped surface experiences gravitational confinement and is sealed off from the spacetime outside the trapped surface (Israel, 1986). The existence of a trapped surface is a sufficient condition for the spacetime to evolve a singularity, given certain assumptions on the spacetime (Hawking and Penrose, 1970). As a result, black hole existence or formation can be predicted by determining whether a trapped surface exists in the spacetime. The advantage of considering trapped surfaces as opposed to event horizons is that trapped surfaces can be studied using only local geometries of the spacetime, which, unlike the global structure, can be fully specified.

One important unsolved problem in the field is the Trapped Surface Conjecture (TSC), which states that on a time slice, if any mass is concentrated in a small enough volume, then there exists a trapped surface containing the mass (Seifert, 1979). A general proof of the TSC would shed light on the conditions necessary for black hole formation. Some progress on the TSC has been made, but with various constraints on the spacetime, such as spherical symmetry, axial symmetry, time symmetry, and the maximal time slice (Khuri, 2009; Malec, 1991; Malec and Xie, 2015; Flanagan, 1991; Schoen and Yau, 1983; Schoen and Yau, 2001). These restrictions limit our understanding of the TSC to only a small subset of possible spacetimes, and no general proof currently exists. A partial proof of the TSC is given in a series of papers by Bizon, Malec, and Murchadha, which present several sufficient conditions and necessary conditions for the existence of a trapped surface that are dependent on the mass content and volume of a region of space (Bizon et al., 1988; Bizon et al., 1989; Bizon et al., 1990). The sufficient conditions in Bizon et al., 1988 are found in the case of a spherically symmetric maximal time slice, and some conditions are obtained with the additional assumption of time symmetry. Therefore, the results of these papers are only applicable to a few cases of spacetimes and must be modified for those without time symmetry, those without a maximal slice, or those that deviate from spherical symmetry.

This paper aims to address these deficits by identifying additional and more general sufficient conditions for the existence of a trapped surface in a spherically symmetric spacetime. Two of these serve as alternatives to the results of Bizon et al., 1988, and one requires weaker assumptions, making the result more generally applicable. In §3, numerical examples of spherically symmetric spacetimes are tested, and it is found that the constant-density star contains trapped surfaces despite not satisfying the main time-symmetric condition of Bizon et al., 1988. In §4, several alternative sufficient conditions for trapped surfaces in a time-symmetric maximal slice are found, one of which correctly predicts trapped surfaces in the constant-density star. In §5, the restriction of the maximal slice is relaxed, leading to a generalization of the main non-time-symmetric result of Bizon et al., 1988 to time slices with non-negative mean curvature in certain regions of the slice. In §6, possible directions for future research are discussed. The results of this paper offer further insight into the validity of the TSC and progress toward the general proof of the TSC.

## 2 Definitions and Preliminaries

In this section, relevant terms are defined and notation to be used throughout this paper is introduced. In a given 4-dimensional spacetime, a *time slice* is a 3-dimensional spacelike hypersurface obtained by holding the timelike coordinate constant.

A time slice is fully described by two pieces of initial data: the *three-metric* $g_{ab}$ and the *extrinsic curvature* $K_{ab}$. From these two, additional quantities can be derived including the three-dimensional *Ricci scalar*, denoted by ${}^{(3)}R$, and the *mean curvature*, equal to the trace of the extrinsic curvature and denoted by $trK = g^{ab}K_{ab}$ (Carroll, 2016).

In some spacetimes there exists a time slice with constant mean curvature, which is called a *CMC slice*. Some spacetimes have a CMC slice where the mean curvature vanishes, in which case the time slice is called *maximal* (Carroll, 2016).

On a given time slice there is a spatial distribution of matter for which geometrical quantities such as mass and volume are defined. The *energy density* at a point is denoted by $\mu$, which can be integrated over a volume to obtain the *mass content* $M$. The *momentum density*, which describes matter flow, is denoted by $j^a$.

Closed two-surfaces can also be defined within the time slice. We use $\delta\Omega$ to denote such a surface and $\Omega$ to denote the enclosed volume. Any $\delta\Omega$ is equipped with a set of future-directed outward null unit normal vectors, which we denote $n^a$. Any $\delta\Omega$ will have a *proper area* (surface area) which we denote $A$. In addition, if $\delta\Omega$ is spherical, we will also have a *proper radius* which we denote $L$. Finally, for any $\delta\Omega$ we can write the enclosed mass content as

$$M = \int_{\Omega} \mu \, dV. \tag{1}$$

To denote covariant derivatives, del notation is used, i.e. the symbol $\nabla_a$ denotes the *covariant derivative operator* with respect to $a$.

Next, several preliminaries are introduced, all of which have previously been established in Bizon et al, 1988. First, properties of matter fields and trapped surfaces in a general spacetime are discussed, then simplifications in the two cases of spherical symmetry and time symmetry are introduced.

**Remark 1.** On any time slice of a given spacetime, there are Hamiltonian constraints on the energy density and momentum density (Bizon et al., 1988):

$$\begin{aligned} {}^{(3)}R - K_{ab}K^{ab} + (trK)^2 &= 16\pi\mu, \\ \nabla_a[K^{ab} - g^{ab}(trK)] &= 8\pi j^b. \end{aligned} \tag{2}$$

**Remark 2.** One definition of a trapped surface is that the rate of expansion of the area of a shell of outgoing future-directed null vectors on the surface must be non-positive. We can write out an expression for this rate of expansion using the initial data on a time slice. A two-surface is trapped if and only if it satisfies (Bizon et al., 1988):

$$\nabla_a n^a + K^{ab} n_a n_b - g_{ab} K^{ab} \leq 0. \tag{3}$$

We refer to (3) as the *expansion condition*.

**Remark 3.** Assume that the time slice is spherically symmetric. Then, in isotropic coordinates, the conformal factor $\Phi$ is a function of the radial coordinate only. We can write the metric in isotropic coordinates as

$$g_{ab} = [\Phi(r)]^4 \delta_{ab} \tag{4}$$

where $\delta_{ab}$ is the flat metric (Bizon et al., 1988). Note that, without loss of generality, we can take $\Phi$ to be always positive. In isotropic coordinates, the Ricci scalar takes the form

$$^{(3)}R = -8\Phi^{-5}\Delta\Phi \tag{5}$$

where $\Delta$ denotes the Laplacian operator in flat coordinates (Bizon et al., 1988). Since $\Phi$ depends only on $r$, using the transformation of a Laplacian from Cartesian coordinates to spherical coordinates, we can rewrite (5) as

$$^{(3)}R = -\frac{8}{r^2}\Phi^{-5}\frac{d}{dr}(r^2\frac{d\Phi}{dr}). \tag{6}$$

**Remark 4.** When the time slice is time-symmetric, the matter is instantaneously at rest and the momentum density vanishes. On a time-symmetric slice the extrinsic curvature $K_{ab}$ vanishes (Bizon et al., 1988); then the expansion condition reduces to

$$\nabla_a n^a \leq 0. \tag{7}$$

In addition, the Hamiltonian constraint for energy density reduces to

$$^{(3)}R = 16\pi\mu. \tag{8}$$

Note that, since the extrinsic curvature of a time-symmetric slice vanishes, the trace of the extrinsic curvature must also vanish, so by definition a time-symmetric slice must also be maximal.

**Remark 5.** Suppose the time slice is both spherically symmetric and time-symmetric. We can foliate the time slice into spherical surfaces of constant coordinate radius on which we test the simplified expansion condition (7). On any such spherical surface, the expansion condition reduces to

$$\frac{d\Phi}{dr} \leq -\frac{\Phi}{2r}. \tag{9}$$

# 3 Conformal Factors and Trapped Surfaces

In this section, we will discuss several specific cases of conformal factors, for which we test both the expansion condition and the main time-symmetric condition in Bizon et al., 1988. Throughout this section we give discussions in spherically symmetric time-symmetric slices.

In Bizon et al., 1988, it is shown that, if

$$M \geq L \tag{10}$$

for a spherical surface in a spherically symmetric, time-symmetric slice, then there must be a trapped surface inside the spherical surface. In the following, we will test the condition $M \geq L$ on several examples of spacetimes.

If a matter distribution is bounded inside some coordinate radius $r'$ and the spacetime is a vacuum outside of the matter distribution region, then at any radius $r \geq r'$ the conformal factor follows the exterior Schwarzschild solution

$$\Phi = 1 + \frac{m}{2r} \tag{11}$$

where $m$ is the ADM mass (Arnowitt et al., 1960) of the matter distribution. Since the Laplacian of (11) vanishes, the Ricci scalar vanishes in the exterior Schwarzschild solution and, by (8), the energy density is zero when $r \geq r'$. We now consider a spherical surface of coordinate radius $r_0 \geq r'$. As we increase the radius of the spherical surface, the mass content remains constant but the proper radius $L$ increases. Therefore, if the condition $M \geq L$ is satisfied for a spherical surface at $r_0 > r'$, then it must be satisfied for a spherical surface at $r_0 = r'$. This indicates that testing $M \geq L$ at the boundary of the matter distribution is sufficient to verify whether a trapped surface exists in the exterior Schwarzschild solution. On the other hand, the expansion condition indicates that a trapped surface exists in the exterior Schwarzschild solution only when

$$r_0 \leq \frac{m}{2}. \tag{12}$$

We see that, for the exterior Schwarzschild solution, the expansion condition predicts trapped surfaces more effectively than $M \geq L$. With the exterior Schwarzschild solution examined, we now consider when a trapped surface occurs inside a bounded matter distribution.

Here we examine a class of conformal factors of the form $\Phi = (1+r^a)^{-b}$ where $a$ and $b$ are positive constants. This class of conformal factors arises as a generalization of the conformal factor for the constant-density star, which is described by $\Phi = (1+r^2)^{-\frac{1}{2}}$. First we test the expansion condition on any $\Phi = (1+r^a)^{-b}$. We find that trapped surfaces appear on spherical surfaces of coordinate radius $r_0$ satisfying

$$r_0 \geq (2ab-1)^{-\frac{1}{a}}. \tag{13}$$

Now we test the condition $M \geq L$. For any $\Phi = (1+r^a)^{-b}$, the Ricci scalar becomes:

$$^{(3)}R = 8ab(1+r^a)^{4b-2} r^{a-2}[(1-ab)r^a + (a+1)]. \tag{14}$$

The weak energy condition requires that the energy density, and therefore the Ricci scalar, is non-negative, so (14) implies a constraint $ab \leq 1$. We now consider a spherical surface $\delta\Omega$ of constant coordinate radius $r_0$. The mass content inside $\delta\Omega$ is

$$\begin{aligned} \int_\Omega \mu \, dV &= \frac{1}{4}\int_0^{r_0} {}^{(3)}R\Phi^6 r^2 \, dr \\ &= 2ab \int_0^{r_0} (1+r^a)^{-2b-2} r^a [(1-ab)r^a + (a+1)] \, dr, \end{aligned} \tag{15}$$

where the first equality follows from using spherical isotropic coordinates. Then the condition $M \geq L$ becomes

$$\int_0^{r_0} (1+r^a)^{-2b-2} 2abr^a [(1-ab)r^a + (a+1)]\, dr \geq \int_0^{r_0} (1+r^a)^{-2b}\, dr. \tag{16}$$

The integral for $M$ contains an additional factor $(1+r^a)^{-2} 2abr^a[(1-ab)r^a + (a+1)]$ compared to the integral for $L$. Note that

$$\lim_{r\to\infty} (1+r^a)^{-2} 2abr^a [(1-ab)r^a + (a+1)] = 2ab(1-ab), \tag{17}$$

and from $ab \leq 1$, the AM-GM inequality gives us $2ab(1-ab) \leq \frac{1}{2}$. Therefore, in (16) the integral for $M$ grows less quickly than the integral for $L$ when $r_0$ becomes large. This implies that $L$ will overtake $M$ at some point and there is an upper bound to the trapped surfaces predicted by the condition $M \geq L$.

Therefore, it is possible that $M \geq L$ is not satisfied anywhere in the spacetime. We note that, in the constant-density star described by $\Phi = (1+r^2)^{-\frac{1}{2}}$, $M \geq L$ becomes

$$\int_0^{r_0} (1+r^2)^{-3} 6r^2\, dr \geq \int_0^{r_0} (1+r^2)^{-1}\, dr, \tag{18}$$

which is not true for any value of $r_0$. Therefore, although (13) indicates that every spherical surface of coordinate radius $r_0 \geq 1$ is trapped for the constant-density star, the condition $M \geq L$ is not satisfied anywhere, so it cannot be used to predict any trapped surfaces in the constant-density star.

Spherically symmetric spacetimes which do not satisfy $M \geq L$ anywhere pose a problem in relation to the TSC. If there does exist a trapped surface somewhere in such a spacetime, as is the case for the constant-density star, we are unable to show, using currently known conditions, that the trapped surface occurs due to a high enough matter concentration in a small enough volume. To address this problem, in the next section, we will find several alternative conditions which state that the mass content must exceed some volume-related quantity, and which can predict trapped surfaces in some spacetimes in which $M \geq L$ is never satisfied.

For the rest of this paper, we will work in time slices which are spherically symmetric.

# 4 Time Symmetry

In this section, we want to find sufficient conditions in spherical symmetry for the existence of a trapped surface in terms of geometrical quantities. The advantage of such conditions is that they will have a direct physical interpretation and, despite our use of isotropic coordinates in deriving them, these conditions will be applicable in any coordinate system. The purpose of this section will be to find alternatives for $M \geq L$ in a time-symmetric slice.

For the rest of this section, we will use spherically symmetric time-symmetric slices, which we recall must also be maximal by §2, Remark 4. We first establish the mass content enclosed in some surface $\delta\Omega$ of coordinate radius $r_0$. Using (5) and (8) and integrating the energy density over the enclosed volume, we have the following equalities for the mass content inside a spherical surface $\delta\Omega$:

$$\begin{aligned} M = \int_\Omega \mu\, dV = \frac{1}{16\pi}\int_\Omega {}^{(3)}R\, dV = \frac{1}{4}\int_0^{r_0} {}^{(3)}R\,\Phi^6 r^2\, dr \\ = -2\int_0^{r_0} \Phi\Delta\Phi r^2\, dr. \end{aligned} \tag{19}$$

We also establish expressions for various geometrical quantities. From the expression for the volume element on a Riemannian manifold (Carroll, 2016), expressions for the proper radius $L$ and proper area $A$ can be derived:

$$\begin{aligned} L &= \int_0^{r_0} \Phi^2\, dr, \\ A &= 4\pi r^2 \Phi^4. \end{aligned} \tag{20}$$

In the following, we will apply (19) and (20) to find several different sufficient conditions for the existence of a trapped surface in the interior of $\delta\Omega$.

**Theorem 1.** *Assume that the time slice is spherically symmetric and time-symmetric. For some* *spherical surface $\delta\Omega$ of coordinate radius $r_0$, if*

$$M \geq \frac{1}{2}L + \sqrt{\frac{A(r_0)}{4\pi}}, \tag{21}$$

*then there exists a trapped surface inside $\delta\Omega$.*

*Proof.* From (19), we have that

$$\begin{aligned}\int_\Omega \mu\, dV &= -2\int_0^{r_0} \Phi\Delta\Phi r^2\, dr \\ &= -2\int_0^{r_0} \Phi\frac{d}{dr}(r^2\frac{d\Phi}{dr})\, dr \\ &= 2\int_0^{r_0} r^2(\frac{d\Phi}{dr})^2\, dr - 2r^2\Phi\frac{d\Phi}{dr}|_{r=r_0}\end{aligned} \tag{22}$$

where we have transformed the Laplacian from flat to spherical coordinates to obtain the second equality and we have integrated by parts to obtain the third equality. The weak energy condition on (5) requires that the Laplacian is non-negative, and since $\Phi$ is positive, $\Phi$ must be a decreasing function of $r$ by the min-max theorem (Bizon et al., 1988).

Assume that there are no trapped surfaces in the interior (including the boundary) of $\delta\Omega$. Then, from (9) we obtain

$$0 < -\frac{d\Phi}{dr} < \frac{\Phi}{2r} \tag{23}$$

everywhere inside $\delta\Omega$. If (23) is true, then from (22) we have an upper bound on the integral of energy density:

$$\begin{aligned}\int_\Omega \mu\, dV &< 2\int_0^{r_0} r^2(\frac{\Phi}{2r})^2\, dr + 2\Phi(r_0)r_0^2\frac{\Phi(r_0)}{2r_0} \\ &= \frac{1}{2}\int_0^{r_0} \Phi^2\, dr + r_0[\Phi(r_0)]^2.\end{aligned} \tag{24}$$

Hence, inequality (24) is a necessary condition for the non-existence of trapped surfaces inside $\delta\Omega$. Note that both terms on the right hand side can be written in terms of the proper radius and proper area. By (24) and (20), if there are no trapped surfaces inside $\delta\Omega$, then we have

$$\int_\Omega \mu\, dV < \frac{1}{2}L + \sqrt{\frac{A(r_0)}{4\pi}}. \tag{25}$$

The theorem follows by contrapositive. ■

Since condition (21) requires the mass content to exceed two terms related to the volume of a spherical surface, Theorem 1 is in agreement with the TSC. Since $L$ and $A$ are independent quantities, Theorem 1 poses a condition which is independent from $M \geq L$. In addition, we find that condition (21) predicts trapped surfaces in some spacetimes that do not satisfy $M \geq L$ anywhere, especially in spacetimes where the proper area of a spherical surface is sufficiently smaller than the proper radius of the surface. For example, we recall from Section 3 that the constant-density star, described by $\Phi = (1+r^2)^{-\frac{1}{2}}$, does not satisfy $M \geq L$ anywhere. On the other hand, for the constant-density star, condition (21) for a spherical surface of coordinate radius $r_0$ becomes

$$6\int_0^{r_0} (1+r^2)^{-3} r^2\, dr \geq \frac{1}{2}\int_0^{r_0} (1+r^2)^{-1}\, dr + r_0(1+r_0^2)^{-1}. \tag{26}$$

This inequality is satisfied when $r_0 \gtrsim 1.764$. By comparison, as derived from the expansion condition in (13), we know that every spherical surface of coordinate radius $r_0 \geq 1$ is trapped in the case of the constant-density star. Therefore, not only does Theorem 1 predict trapped surfaces in this case where $M \geq L$ is not satisfied anywhere in the spacetime, but it also predicts trapped surfaces for a range of $r_0$ unbounded from above, which agrees with our calculation in (13). We see that Theorem 1 proposes a viable alternative condition to $M \geq L$.

Next, we note that the energy density $\mu$ depends only on $\Phi$, so we can consider $\mu$ as a function of $r$, i.e., $\mu = \mu(r)$. In the following theorem, we derive a different sufficient condition for trapped surfaces that involves the derivative of $\mu(r)$.

**Theorem 2.** *Assume that the time slice is spherically symmetric and time-symmetric. For some spherical surface $\delta\Omega$ of coordinate radius $r_0$, if*

$$\int_0^{r_0} A^{\frac{3}{2}} \frac{d\mu}{dr} dr \geq [A(r_0)]^{\frac{3}{2}} \mu(r_0), \tag{27}$$

*then there exists a trapped surface inside $\delta\Omega$.*

*Proof.* From (6) and (8), we can find an expression for the derivative of the Laplacian of the conformal factor:

$$\frac{d}{dr}(\Delta\Phi) = -2\pi\frac{d\mu}{dr}\Phi^5 + 5\Phi^{-1}(\frac{d\Phi}{dr})\Delta\Phi. \tag{28}$$

We begin from (19) and obtain:

$$\begin{aligned}\int_\Omega \mu\, dV &= -2\int_0^{r_0} r^2\Phi\Delta\Phi\, dr \\ &= -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + \frac{2}{3}\int_0^{r_0} r^3\frac{d}{dr}(\Phi\Delta\Phi)\, dr \\ &= -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + \frac{2}{3}\int_0^{r_0} r^3(\frac{d\Phi}{dr}\Delta\Phi + \Phi\frac{d}{dr}(\Delta\Phi))\, dr,\end{aligned} \tag{29}$$

where the second equality is obtained through integration by parts. Substituting (28) into (29), we obtain:

$$\begin{aligned}\int_\Omega \mu\, dV &= -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + \frac{2}{3}\int_0^{r_0} r^3[\frac{d\Phi}{dr}\Delta\Phi + \Phi(-2\pi\frac{d\mu}{dr}\Phi^5 + 5\Phi^{-1}\frac{d\Phi}{dr}\Delta\Phi)]\, dr \\ &= -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + 4\int_0^{r_0} r^3\frac{d\Phi}{dr}\Delta\Phi\, dr - \frac{4}{3}\pi\int_0^{r_0} r^3\Phi^6\frac{d\mu}{dr}\, dr.\end{aligned} \tag{30}$$

Assume there are no trapped surfaces inside $\delta\Omega$. Then $0 > \frac{d\Phi}{dr} > -\frac{\Phi}{2r}$ everywhere inside $\delta\Omega$, and since $\Delta\Phi$ is negative, we have an upper bound on the integral of energy density:

$$\begin{aligned}\int_\Omega \mu\, dV &< -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + 4\int_0^{r_0} r^3(-\frac{\Phi}{2r})\Delta\Phi\, dr - \frac{4}{3}\pi\int_0^{r_0} r^3\Phi^6\frac{d\mu}{dr}\, dr \\ &= -\frac{2}{3}r^3\Phi\Delta\Phi|_{r=r_0} + \int_\Omega \mu\, dV - \frac{4}{3}\pi\int_0^{r_0} r^3\Phi^6\frac{d\mu}{dr}\, dr.\end{aligned} \tag{31}$$

where we use (19) to obtain the first equality. We see that the integrals of energy density on the left hand side and right hand side cancel. By contrapositive, if

$$\int_0^{r_0} r^3\Phi^6\frac{d\mu}{dr}\, dr \geq -\frac{1}{2\pi}r^3\Phi\Delta\Phi|_{r=r_0} = r_0^3[\Phi(r_0)]^6\mu(r_0), \tag{32}$$

then there exists a trapped surface inside $\delta\Omega$. Substituting the proper area $A = 4\pi r^2\Phi^4$ into (32) yields the theorem. ■

While condition (27) uses the radial coordinate, it depends only on terms derived from two geometrical quantities, the proper area and energy density, so it is applicable in any spherically symmetric coordinate system rather than only the isotropic coordinate system. In this respect, it holds an advantage over the expansion condition, which, in a general coordinate system, can only be specified in terms of the initial data rather than geometrical quantities.

However, there are several aspects of condition (27) that limit its usefulness. Condition (27) includes an upper bound on a term proportional to the energy density at a point. As a result, it deviates from our expectation that the mass content must exceed some volume-related quantity of the spherical surface for a trapped surface to exist, as stated in the TSC. In addition, in spacetimes where $\mu(r)$ is decreasing, the integral on the left hand side will be non-positive but the quantity on the right hand side is strictly positive, so (27) is not satisfied anywhere in such spacetimes.

# 5 Nonvanishing Extrinsic Curvature

In this section, we will remove the constraint of the time-symmetric slice, on which the extrinsic curvature vanishes, and we allow for matter flow, i.e nonzero $j^b$. In Bizon et al., 1988, the authors discuss a matter distribution in a maximal slice, where the mean curvature vanishes, and obtain a sufficient condition for the existence of trapped surfaces based on both mass content and matter flow:

$$\int_\Omega \mu - j{\cdot}n\,dV \geq \frac{7}{6}L. \tag{33}$$

However, the maximal slice is a restrictive assumption that may not be found in every spherically symmetric spacetime. A stronger result may be found by relaxing restrictions on the mean curvature. In the following theorem, we show that condition (33) is applicable in spacetimes with a much weaker set of assumptions.

**Theorem 3.** *Let $\delta\Omega$ be a spherical surface in a spherically symmetric time slice. If*

$$trK \geq 0 \tag{34}$$

*everywhere inside $\delta\Omega$ and*

$$\int_\Omega \mu - j{\cdot}n\,dV \geq \frac{7}{6}L, \tag{35}$$

*then there exists a trapped surface inside $\delta\Omega$.*

*Proof.* The Hamiltonian constraints and expansion condition both depend on several scalars derived from the extrinsic curvature, including $K_{ab}K^{ab}$, $K^{ab}n_a n_b$. In addition, the term $K^{ab}\nabla_b n_a$ appears in the integral of the Hamiltonian constraint for momentum density. To directly calculate these scalars, we require an explicit formula for the extrinsic curvature. In a spherically symmetric time slice, we can always write the extrinsic curvature as

$$K_{ab} = (n_a n_b)K_L + (g_{ab} - n_a n_b)K_R, \tag{36}$$

where $K_L$ and $K_R$ are both functions of the radial coordinate only (Guven and Murchadha, 1999). Contracting (36), we obtain

$$\begin{aligned} trK = g^{ab}K_{ab} &= (g^{ab}n_a n_b)K_L + (g^{ab}g_{ab})K_R - (g^{ab}n_a n_b)K_R \\ &= K_L + 2K_R. \end{aligned} \tag{37}$$

This allows us to rewrite (36) in terms of $trK$ and $K_R$:

$$\begin{aligned} K_{ab} &= (n_a n_b)trK + (g_{ab} - 3n_a n_b)K_R \\ &= (n_a n_b)trK + (n_a n_b - \frac{1}{3}g_{ab})K' \end{aligned} \tag{38}$$

where $K' = -3K_R$. Now we raise the indices of the extrinsic curvature to obtain $K^{ab}$ and calculate the scalars $K_{ab}K^{ab}$, $K^{ab}n_a n_b$, and $K^{ab}\nabla_b n_a$ as below:

$$\begin{aligned} K_{ab}K^{ab} &= (trK)^2 + \frac{4}{3}(trK)K' + \frac{2}{3}K'^2, \\ K^{ab}n_a n_b &= trK + \frac{2}{3}K', \\ K^{ab}\nabla_b n_a &= -\frac{1}{3}K'\Phi^{-6}r^{-2}\frac{d}{dr}(\Phi^4 r^2). \end{aligned} \tag{39}$$

Note that $K^{ab}\nabla_b n_a$ does not depend on $trK$, so the value of $K^{ab}\nabla_b n_a$ in any CMC slice is identical to that in the maximal slice. Then the Hamiltonian constraints (2) become

$$\begin{aligned} -8\Phi^{-5}\Delta\Phi - \frac{4}{3}(trK)K' - \frac{2}{3}K'^2 = 16\pi\mu, \\ \nabla_a K^{ab} - g^{ab}\nabla_a(trK) = 8\pi j^b. \end{aligned} \tag{40}$$

The expansion condition (3) becomes

$$\frac{d}{dr}(r\Phi^2)+\frac{1}{3}K'r\Phi^4 \le 0. \tag{41}$$

Note that, in the expansion condition, all the terms involving $trK$ cancel, so the expansion condition in a general time slice are unchanged compared to those in the maximal slice. Using the scalars derived from $K_{ab}$, based on the reasoning in Bizon et al., 1988, we write out the integral $\int_\Omega \mu - j{\cdot}n\,dV$ with a few extra terms involving $trK$:

$$\begin{aligned}16\pi\int_\Omega \mu - j{\cdot}n\,dV = &-[16\pi r\frac{d}{dr}(r\Phi^2)]|_{r=r_0}+16\pi\int_0^{r_0}[\frac{d}{dr}(r\Phi^2)+2r^2(\frac{d\Phi}{dr})^2]\,dr\\ &-\frac{16}{3}\pi\int_0^{r_0}(trK)K'r^2\Phi^6\,dr-\frac{8}{3}\pi\int_0^{r_0}K'^2r^2\Phi^6\,dr\\ &-[8\pi r^2\Phi^4(trK+\frac{2}{3}K')]|_{r=r_0}-\frac{8}{3}\pi\int_0^{r_0}K'\frac{d}{dr}(r^2\Phi^4)\,dr\\ &+8\pi\int_0^{r_0}r^2\Phi^4\frac{d}{dr}(trK)\,dr.\end{aligned} \tag{42}$$

Using (41), if there are no trapped surfaces inside $\delta\Omega$, then everywhere throughout $\Omega$ we have

$$\frac{d}{dr}(r\Phi^2)+\frac{1}{3}K'r\Phi^4 > 0. \tag{43}$$

It was established in Bizon et al., 1988 that, in the case of no trapped surfaces, we have a bound

$$\begin{aligned}16\pi\cdot\frac{7}{6}L > &-[16\pi r\frac{d}{dr}(r\Phi^2)]|_{r=r_0}+16\pi\int_0^{r_0}[\frac{d}{dr}(r\Phi^2)+2r^2(\frac{d\Phi}{dr})^2]\,dr\\ &-\frac{8}{3}\pi\int_0^{r_0}K'^2r^2\Phi^6\,dr-[\frac{16}{3}\pi r^2\Phi^4K']|_{r=r_0}-\frac{8}{3}\pi\int_0^{r_0}K'\frac{d}{dr}(r^2\Phi^4)\,dr.\end{aligned} \tag{44}$$

Assume that there are no trapped surfaces inside $\delta\Omega$. Then, we have

$$
\begin{aligned}
16\pi \int_\Omega & \mu - j{\cdot}n\, dV \\
& < 16\pi \cdot \frac{7}{6}L - [8\pi r^2 \Phi^4 trK]|_{r=r_0} - \frac{16}{3}\pi \int_0^{r_0} (trK)K' r^2 \Phi^6\, dr \\
& \quad + 8\pi \int_0^{r_0} r^2 \Phi^4 \frac{d}{dr}(trK)\, dr \\
& = 16\pi \cdot \frac{7}{6}L - 8\pi \int_0^{r_0} r^2 \Phi^4 \frac{d}{dr}(trK)\, dr - 16\pi \int_0^{r_0} (trK) r\Phi^2 \frac{d}{dr}(r\Phi^2)\, dr \\
& \quad - \frac{16}{3}\pi \int_0^{r_0} (trK)K' r^2 \Phi^6\, dr + 8\pi \int_0^{r_0} r^2 \Phi^4 \frac{d}{dr}(trK)\, dr \\
& = 16\pi \cdot \frac{7}{6}L - 16\pi \int_0^{r_0} (trK) r\Phi^2 [\frac{d}{dr}(r\Phi^2) + \frac{1}{3}K' r\Phi^4]\, dr,
\end{aligned}
\tag{45}
$$

where we have applied integration by parts and the identity $\frac{d}{dr}(r^2\Phi^4) = 2r\Phi^2 \frac{d}{dr}(r\Phi^2)$ to obtain the first equality. If $trK \geq 0$ and there are no trapped surfaces inside $\delta\Omega$, then (43) is true and the second term in (45) is always negative, so we obtain

$$
16\pi \int_\Omega \mu - j{\cdot}n\, dV < 16\pi \cdot \frac{7}{6}L. \tag{46}
$$

The theorem follows by contrapositive. ■

Theorem 3 indicates that it is not necessary for a spherically symmetric spacetime to have a maximal slice for condition (33) to be applicable. The geometric interpretation of positive mean curvature inside $\delta\Omega$ is that every region inside $\delta\Omega$ locally increases in area in the direction of the normal to the time slice with respect to which we have calculated the mean curvature.

One setting in which condition (33) is useful is a CMC slice, which is often found in cosmologies or asymptotically flat spherically symmetric spacetimes (Iriondo et al., 1996). Condition (33) can be applied in any spacetime with a CMC slice for which the mean curvature is non-negative with respect to the future-pointing normal to the time slice. However, we notice that the negative second term in (45) is proportional to $trK$, which implies that (33) becomes a stricter sufficient condition, and therefore a poorer predictor for trapped surfaces, at higher values of the mean curvature. Condition (33) is most effective at predicting trapped surfaces in a maximal slice or in a near-maximal slice, in which the mean curvature takes a very small value.

Note that, when $trK = 0$, Theorem 3 reduces to the main non-time-symmetric result of Bizon et al., 1988.

# 6 Future Research

The results found in this paper rely on perfect spherical symmetry, which is uncommon in real-world stellar matter configurations. One modification to the spacetime that may better model physical scenarios is a small perturbation of spherical symmetry in a conformally flat spacetime. In such a spacetime, the conformal factor $\Phi$ takes the form

$$\Phi = f(r) + \varepsilon f'(x^1, x^2, x^3) \tag{47}$$

where $r$ is the radial coordinate, $x^1, x^2, x^3$ are the three spacelike coordinates, and the constant $\varepsilon > 0$ is very small. In the case of a small perturbation of spherical symmetry, similar conditions relating mass content and volume-related quantities may be found that are similar to those in Bizon et al., 1988 and in this paper but with some modifications of the terms.

The case of axial symmetry corresponds to matter configurations with angular momentum, so axial symmetry is a more accurate model of physical scenarios compared to perfect spherical symmetry. Several inequalities relating mass, angular momentum, and volume-related quantities for axisymmetric black holes have been found (Dain, 2014; Hennig et al., 2008; Dain, 2008). It has been found that sufficient concentration of angular momentum in a small volume can lead to the formation of a black hole (Khuri, 2015). Some progress on the TSC in the axisymmetric case has been made with the use of Brill coordinates (Khuri and Xie, 2017), in which the line element takes the form:

$$ds^2 = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2. \tag{48}$$

In this metric the Ricci scalar takes the form below (Chruściel, 2008):

$$-\frac{e^{-2U+2\alpha}}{4}{}^{(3)}R = -\Delta(U - \frac{1}{2}\alpha) + \frac{1}{2}(DU)^2 - \frac{1}{2\rho}\frac{\partial \alpha}{\partial \rho} + \frac{\rho^2 e^{-2\alpha}}{8}(\partial_\rho A_z - \partial_z A_\rho)^2, \tag{49}$$

where $\Delta$ and $D$ are the flat-space Laplacian and gradient, respectively. In the constraint of time symmetry, the Ricci scalar is directly proportional to the energy density at a point, greatly simplifying calculations of mass content inside a closed two-surface using the axisymmetric Brill coordinates. Using these techniques, further sufficient conditions and necessary conditions relating mass content, angular momentum, and volume or area may be found for the existence of a trapped surface in an axisymmetric spacetime.

It has also been claimed that concentration of gravitational radiation can lead to trapped surface formation. In Beig and Murchadha, 1991, the authors constructed initial data with zero matter-energy and showed that trapped surfaces form in cases when certain global invariants are very small. Further evidence may be found for the existence of trapped surfaces due to concentration of gravitational radiation that may shed light on novel extensions of the TSC.

# 7 Conclusion

In this paper, we found sufficient conditions for the existence of a trapped surface inside a closed spherical two-surface on a time-symmetric time slice of a spherically symmetric spacetime. For a given matter configuration, if the mass content $M$ inside a spherical two-surface satisfies $M \geq \frac{1}{2}L + \sqrt{\frac{A(r_0)}{4\pi}}$, then the two-surface contains a trapped surface. This condition requires that the mass content exceed a volume-related quantity of the two-surface, providing further evidence for the validity of the Trapped Surface Conjecture. We found an additional sufficient condition for the existence of a trapped surface relating the proper area, energy density, and derivative of energy density with respect to the radial coordinate. We also generalized the main non-time-symmetric condition of Bizon et al., 1988, which relates mass content, radial flow, and proper radius, to spacetimes that do not contain a maximal slice. We also suggested avenues for further progress on the Trapped Surface Conjecture, including small perturbations of spherically symmetric spacetimes as well as the time-symmetric axisymmetric case in Brill coordinates. Our results, as well as the future directions if pursued, constitute progress toward the realization of the Trapped Surface Conjecture, offering insights into the conditions surrounding black hole formation that are of use to both mathematicians and cosmologists.

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